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A DIMENSION THEOREM FOR DIVISION RINGS

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ABSTRACT

Let D be a division algebra over a field k, let n be an arbitrary positive integer, and let $k(x_1,\cdots,x_n)$ denote the rational function field in n variables over k. In this note we complete previous work by proving that the following three conditions are equivalent: (i) there exists an integer j such that the matrix ring $M_j(D)$ contains a commutative subfield which has transcendence degree n over k; (ii) K $\dim(D\otimes_k k(x_1,\cdots,x_n))=n$; (iii) gl. $\dim(D\otimes_k k(x_1,\cdots,x_n))=n$. The crucial tool in the proof of this theorem is the Nullstellensatz for $D[x_1,\cdots,x_n]$ which was obtained by Amitsur and Small.

Let A be an algebra over a field k. In [6; definition 3.1] we defined the transcendence degree of A over k, written tr. deg. (A/k), as the supremum of the transcendence degrees of all the commutative subalgebras of A, and showed [6; 3.15, 3.17] that if A is semisimple Artinian and has tr. deg. $(A/k)\geqq n$ then the algebra $A\otimes_k k(x_1,\cdots,x_n)$ has (left or right) Krull dimension and global dimension equal to n. If D is a division algebra over k and $n=1$, then the conditions

$$\text{tr. deg.}(M_j(D)/k)\geqq n,\quad \text{K}\dim(D\otimes k(x_1,\cdots,x_n))=n,$$

$$\text{gl.}\dim(D\otimes k(x_1,\cdots,x_n))=n$$

all reduce to the statement that $D\otimes k(x)$ is not a division ring, hence are equivalent. It was an open question, however, whether the same assertion could be made for arbitrary n. In this note we shall resolve this question by proving the following theorem.

THEOREM. *Let D be a division algebra over the field k, let n be an arbitrary positive integer, and let $k(x_1,\cdots,x_n)$ denote the rational function field in n*

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variables over k. Then the following are equivalent:

(i) *there exists an integer j such that the transcendence degree of the matrix ring $M_j(D)$ over k is at least n;*

(ii) $\operatorname{K dim}(D \otimes_k k(x_1, \cdots, x_n)) = n$;

(iii) $\operatorname{gl.dim}(D \otimes_k k(x_1, \cdots, x_n)) = n$.

When k is the center of the division ring D, we can add a fourth condition to our list of equivalences: the polynomial ring $D[x_1, \cdots, x_n]$ is primitive [1; theorem 2]. Note also that condition (i) of the theorem is left-right symmetric. Thus, the left and right dimensions of $D \otimes k(x_1, \cdots, x_n)$ are always equal and the ring $D[x_1, \cdots, x_n]$ is left primitive iff right primitive.

To prove the theorem it will be enough, by the results quoted in the first paragraph, to establish that (ii) $\Rightarrow$ (i), (iii) $\Rightarrow$ (i). Our proof of these implications will make heavy use of the Amitsur–Small Nullstellensatz [1; theorem 1]: If D is a division ring, then simple $D[x_1, \cdots, x_n]$-modules are finite dimensional over D. We isolate the necessary technical preliminaries as Lemma 1 and Lemma 3, respectively. For the definition of a *right denominator set* in a ring R, we refer the reader to [10; pp. 51–52]. If S is such a set, we let R_S denote the right ring of fractions, while if M is an R-module we let $M_S = M \otimes_R R_S$ denote the module of fractions [10; p. 57]. Unless explicitly stated otherwise, all modules are right modules and all one-sided conditions are taken on the right. With regard to Lemma 3, the author would like to thank Lance Small for bringing the extremely useful results of [2] and [9] to his attention.

Lemma 1. *Let R be a right Noetherian ring of finite Krull dimension and let $S \subset R$ be a right denominator set. If the rings R and R_S have the same Krull dimension, then some simple R-module is S-torsion free.*

Proof. The proof depends on the following observation, for which the Noetherian hypothesis is superfluous: If M is a nonzero R-module of finite Krull dimension and $\operatorname{K dim}_{R_S}(M_S) = \operatorname{K dim}_R(M)$, then for every integer $0 \leqq j \leqq \operatorname{K dim}_R(M)$ there exists a subquotient (that is, homomorphic image of a submodule) M_j of M with $\operatorname{K dim}_{R_S}(M_j \otimes_R R_S) = j = \operatorname{K dim}_R(M_j)$.

The proof is by induction on $n = \operatorname{K dim}_R(M)$. For $n = 0$ there is nothing to prove. So assume that $n > 0$ and that the proposition holds for all R-modules of Krull dimension less than n. Since $\operatorname{K dim}_{R_S}(M_S) = n > 0$, there exists an infinite descending chain $M_S = N_0 \supset N_1 \supset N_2 \supset \cdots$ of R_S-submodules of M_S with $\operatorname{K dim}_{R_S}(N_i/N_{i+1}) \geqq n-1$. If $\mu : M \to M \otimes_R R_S$ is defined by $\mu(m) = m \otimes 1$, then the map $N \to \mu^{-1}(N)$ is an injection of the set of R_S-submodules of M_S into the set of R-submodules of M whose left inverse is the map $L \to \mu(L)$ [10; p. 62]. It follows that $\operatorname{K\,dim}_{R_S}(K_S) \leqq \operatorname{K\,dim}_R(K)$ for any R-module K. In particular, if $L_i = \mu^{-1}(N_i)$ then $L_i \otimes_R R_S \simeq N_i$ and

$$\operatorname{K\,dim}_R(L_i/L_{i+1}) \geqq \operatorname{K\,dim}_{R_S}(N_i/N_{i+1}) \geqq n-1.$$

Since the Krull dimensions of both M and M_S are n, we must have

$$\operatorname{K\,dim}_R(L_i/L_{i+1}) = n-1 = \operatorname{K\,dim}_{R_S}(N_i/N_{i+1})$$

for all i greater than or equal to some sufficiently large integer i_0. Putting $M' = L_{i_0}/L_{i_0+1}$, we obtain a subquotient of M with $\operatorname{K\,dim}_{R_S}(M' \otimes_R R_S) = n-1 = \operatorname{K\,dim}_R(M')$. By induction for each $j < n-1$ there exists a subquotient M_j of M', hence of M, with $\operatorname{K\,dim}_{R_S}(M_j \otimes_R R_S) = j = \operatorname{K\,dim}_R(M_j)$.

Now suppose that R is right Noetherian of finite Krull dimension and that $\operatorname{K\,dim} R_S = \operatorname{K\,dim} R$. By the preceding paragraph we can find right ideals $I \subset J \subset R$ such that $\operatorname{K\,dim}_R(J/I) = 0 = \operatorname{K\,dim}_{R_S}(J_S/I_S)$. Being Noetherian, J/I has finite length. If $J = J_0 \supset \cdots \supset J_m = I$ is a composition series for J/I, then the fact that $I_S \neq J_S$ implies that $(J_i/J_{i+1})_S \neq 0$ for at least one simple composition factor J_i/J_{i+1}. □

For the proof of Lemma 3 we will need the following proposition which is due to P. Smith [9].

Lemma 2. *Let R be a left and right Noetherian ring and let $U \subset R$ be a two-sided ideal which is generated by central elements. If $T = \{1-u \mid u \in U\}$, then T is a left and a right denominator set in R.*

Proof. Given that U is centrally generated, the usual commutative argument can be adapted to show that U has the Artin–Rees property [4; p. 486]. By [9; theorem 1.1] the set T satisfies the right Ore condition. As R is right Noetherian, this is enough to guarantee that T is a right denominator set [10; p. 52]. Because R is left Noetherian, a dual argument shows that T is also a left denominator set. □

We let $\operatorname{pd}_R(M)$ (respectively, $\operatorname{wd}_R(M)$) denote the projective (weak) dimension of an R-module M.

Lemma 3. *Let R be a left and right Noetherian ring of finite global dimension d, and let S be a multiplicatively closed subset of central, regular elements in R. Then the rings R and R_S have the same right global dimension iff there exists a simple right R-module M which has $\operatorname{pd}_R(M) = d$ and which is S-torsion free.*

Proof. ($\Rightarrow$) Suppose $\operatorname{gl.dim}(R_S) = \operatorname{gl.dim}(R) = d$. By Bhatwadekar's theorem [2; proposition 1.1] there exists a maximal right ideal I' of R_S with $\mathrm{pd}_{R_S}(R_S/I') = d$. If $I = I' \cap R$, then clearly R/I is S-torsion free and $\mathrm{pd}_R(R/I) = d$. We claim that I is a maximal right ideal in the ring R. The proof consists in showing that if there exists any proper right ideal of R which properly contains I, there exists a right ideal J with this property for which $\mathrm{pd}_R(R/J) = d + 1$.

So let $J \supset I$ be such a right ideal. Since $I_S = I'$ is maximal in R_S, $J \cap S \neq \varnothing$. For our purposes it is enough to consider the case $J = I + cR$, where $c \in S$. Let $T = \{1 - cr \mid r \in R\}$. As c is central, T is a right denominator set by Lemma 2. Because R/I has no S-torsion, we have an exact sequence of R-modules

$$0 \to R/I \xrightarrow{c} R/I \to R/J \to 0,$$

hence an exact sequence of R_T-modules

$$0 \to R_T/I_T \xrightarrow{c} R_T/I_T \to R_T/J_T \to 0.$$

The element c is manifestly regular on both R_T and R_T/I_T. On the other hand, if $R_T/J_T = 0$ then $J \cap T \neq \varnothing$. From $j = 1 - cr \in T$, we get $1 = j + cr \in J$, which contradicts our assumption that J is proper. The definition of T puts c in the Jacobson radical of R_T and from [3; pp. 178–179] we conclude that $\mathrm{pd}_{R_T}(R_T/J_T) = 1 + \mathrm{pd}_{R_T}(R_T/I_T)$.

To complete the argument it is enough to show that $\mathrm{pd}_{R_T}(R_T/I_T) = \mathrm{pd}_R(R/I)$. Since R_T is right Noetherian, it is well-known that $\mathrm{pd}_{R_T}(N) = \mathrm{wd}_{R_T}(N)$ for any finitely generated R_T-module N (this follows from [10; p. 30], for example). Further, since T is both a right and a left denominator set, R_T is both a left flat and a right flat extension of R. This, together with the isomorphism $R_T \otimes_R R_T \simeq R_T$, implies that $\mathrm{wd}_{R_T}(M_T) = \mathrm{wd}_R(M_T)$ for any R-module M. Now note that R/I is T-torsion free. For let $K \supset I$ be the preimage of the T-torsion submodule of R/I. If $I \neq K$, then there exists $b \in S \cap K$. But $bt = tb \in I$ forces $t = (tb)b^{-1} \in I_S \cap R = I$, whence we deduce $1 \in J$, again contradicting the properness of J. We thus obtain an exact sequence of R-modules

$$0 \to R/I \xrightarrow{\mu} R/I \otimes_R R_T \to Q \to 0,$$

where μ is the canonical map and $Q = \operatorname{Coker} \mu$. Inasmuch as $\mathrm{wd}_R(R/I) = n = \mathrm{gl.dim}(R)$, we must have $\mathrm{wd}_R(R_T/I_T) = n$. It follows that $\mathrm{pd}_R(R/J) = d + 1$ and that the assumption $J \neq R$ is untenable.

($\Leftarrow$) Let M be a finitely generated R-module which has $\mathrm{pd}_R(M) = d$ and which is S-torsion free. As S is both a left and a right denominator set in R, we can repeat the argument of the preceding paragraph to conclude that

$\mathrm{wd}_{R_S}(M \otimes_R R_S) = \mathrm{wd}_R(M \otimes_R R_S) = d$. This gives $\mathrm{gl.dim}(R_S) \geqq d$, and the reverse inequality is standard [3; p. 181]. □

Before proceeding to the proof of the dimension theorem, we note an application of Lemma 3 to a closely related, but somewhat more general, problem. If k is a field and A is a k-algebra, then one has the estimates

$$\mathrm{gl.dim}(A) \leqq \mathrm{gl.dim}(A \otimes_k k(x_1, \cdots, x_n) \leqq \mathrm{gl.dim}(A) + n,$$

and it is natural to ask when the upper bound in this inequality is attained. In [7; corollary 1.7] Rinehart and Rosenberg proved that if R is a right Noetherian ring and M is a right $R[x]$-module which is finitely generated and has finite projective dimension as an R-module, then $\mathrm{pd}_{R[x]}(M) = \mathrm{pd}_R(M) + 1$. Their result, in conjunction with Lemma 3, will enable us to answer the above question for left and right Noetherian algebras of finite global dimension.

PROPOSITION 1. *Let k be a field and let A be a left and right Noetherian k-algebra of finite global dimension d. Then* $\mathrm{gl.dim}(A \otimes_k k(x_1, \cdots, x_n)) = d + n$ *iff there exists a finitely generated A-module M with the properties that* $\mathrm{pd}_A(M) = d$ *and that* $\mathrm{End}_A(M)$ *contains a commutative subfield of transcendence degree n over k.*

PROOF. To simplify notation, put $R = A[x_1, \cdots, x_n]$, $S = k[x_1, \cdots, x_n] - \{0\}$. It is clear that S is a multiplicative subset of R satisfying the hypotheses of Lemma 3, and that $A \otimes k(x_1, \cdots, x_n) \simeq R_S$.

($\Rightarrow$) Suppose that $\mathrm{gl.dim}(R_S) = d + n$. Then by [3; p. 174] and Lemma 3 there exists a simple R-module M which is S-torsion free and which has $\mathrm{pd}_R(M) = d + n$. If follows from [8; theorem 3.8] that M is finitely generated as an A-module, and it follows from [7; corollary 1.7] that $\mathrm{pd}_A(M) = d$. Now $\Delta = \mathrm{End}_R(M)$ is a division ring, and the embedding $k[x_1, \cdots, x_n] \to \Delta$ can be extended to $k(x_1, \cdots, x_n)$, the quotient field of $k[x_1, \cdots, x_n]$. Since Δ obviously embeds in $\mathrm{End}_A(M)$, M is an A-module with the required properties.

($\Leftarrow$) Let M be an A-module which satisfies the given conditions and let $\{\theta_1, \cdots, \theta_n\}$ be a commuting, algebraically independent subset of $\mathrm{End}_A(M)$. Make M an R-module by letting x_i act like θ_i. Since M is finitely generated over A, the result of Rinehart–Rosenberg implies that $\mathrm{pd}_R(M) = d + n$. Inasmuch as the nonzero elements of the subalgebra $k[\theta_1, \cdots, \theta_n]$ of $\mathrm{End}_A(M)$ act like automorphisms of M, $M \simeq M_S$ is S-torsion free. Arguing as in the last paragraph of the proof of Lemma 3, we get $\mathrm{pd}_{R_S}(M_S) = d + n$, whence $\mathrm{gl.dim}(A \otimes k(x_1, \cdots, x_n)) = d + n$. □

The result of Rinehart and Rosenberg admits another application of interest.

PROPOSITION 2. *Let D be a division ring. Then any $D[x_1, \cdots, x_n]$-module of finite length has projective dimension n.*

PROOF. The proof is by induction on n, the number of variables. For $n = 1$ the result is trivial — any $D[x]$-module of finite length is necessarily torsion with respect to the regular elements of $D[x]$ and cannot be projective. So assume $n > 1$ and put $R = D[x_1, \cdots, x_{n-1}]$. If M is an $R[x]$-module of finite length, then it follows easily from [1; theorem 1] that M_R, the R-module obtained by restriction of scalars, also has finite length. By induction $\mathrm{pd}_R(M) = n - 1$, and by [7; corollary 1.7] $\mathrm{pd}_{R[x]}(M) = n$. □

We return to the proof of the dimension theorem.

PROOF OF THEOREM. (i) $\Rightarrow$ (ii). By [6; theorem 3.15] and [5; application 1] the ring $M_j(D) \otimes_k k(x_1, \cdots, x_n)$ has Krull dimension n. Since

$$M_j(D) \otimes k(x_1, \cdots, x_n) \simeq M_j(D \otimes k(x_1, \cdots, x_n)),$$

the basic properties of Krull dimension show that $\mathrm{K}\dim(D \otimes k(x_1, \cdots, x_n))$ is also n.

(ii) $\Leftarrow$ (i). As in Proposition 1, we put $R = D[x_1, \cdots, x_n]$, $S = k[x_1, \cdots, x_n] - \{0\}$. By Lemma 1 there exists a maximal right ideal $I \subset R$ such that $I \cap k[x_1, \cdots, x_n] = 0$. By the Amitsur–Small Nullstellensatz the factor $M = R/I$ has finite dimension as a vector space over D, say $\dim_D(M) = j$. The obvious embedding $k[x_1, \cdots, x_n] \rightarrow M_j(D)$ shows that $\mathrm{tr.deg.}(M_j(D)/k) \geqq n$.

(i) $\Leftrightarrow$ (iii). This is immediate from Proposition 1. □

We close with a few remarks concerning the relevance of the dimension theorem to a question raised in [1; p. 358] and [6; 4.10]: If some matrix ring $M_j(D)$ has transcendence degree n over k, does it follow that D itself has transcendence degree n over k? This is the question of whether the transcendence degree of a division algebra is a Morita invariant, and is the natural generalization of a longstanding open problem in ring theory. As the results of [6; section 4] show, the dimension properties of $D \otimes k(x_1, \cdots, x_n)$ can frequently be used to obtain an affirmative answer to this question. We had hoped that similar indirect methods might yield a solution to the general problem, but, in view of the theorem proven here, this now seems highly unlikely. The most that one can conclude from the theorem is that if there exists an integer n such that the dimension of $D \otimes k(x_1, \cdots, x_{n+1})$ is strictly less than $n + 1$, then the transcendence degree of D and all its matrix rings is at most n.

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